Primary scalar hair in dilatonic theories with modulus fields

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We study the general spherical symmetric solutions of dilaton-modulus gravity nonminimally coupled to a Maxwell field, using methods from the theory of dynamical systems. We show that the solutions can be classified by the mass, the magnetic charge, and a third parameter which we argue can be related to a scalar charge. The global properties of the solutions are discussed.

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INTRODUCTION

General relativity is a very successful theory in describing classical gravity. However, if one wants to investigate the relations between gravity and quantum mechanics, the frame-work of general relativity is no longer sufficient, and one has to resort to a more general theory. Among the numerous attempts in this direction, one of the most successful at present seems to be string theory. Although its implications for quantum gravity cannot be discussed in full at the present level of development of the theory, it is however interesting to notice that some effects are present already at the classical level, since the effective gravitational action derived from the low-energy approximation of string theory is different from the usual Einstein-Hilbert action. In first approximation, the difference is due to the presence of extra scalar fields, such as the dilaton and the moduli, which are nonminimally coupled with gravity and other fields. The coupling with these scalars modifies some of the classical features of general relativity, such as, for example, black hole thermodynamics [1–3]. It is therefore of great importance to investigate in detail if other standard properties of general relativity are also affected.

A remarkable example is the classification of black hole solutions. It is known, in fact, that black hole solutions of charged dilaton-gravity models, such as those arising in effective string theory, present quite different properties from the Reissner-Nordström solutions of general relativity [1–2]. This is caused essentially by the nonminimal coupling of the dilaton to the Maxwell field, which does not permit the application of the standard no-hair theorems [4] and hence allows for the presence of a nontrivial dilaton field. In spite of this, the dilaton charge is not an independent parameter, but is still a function of the mass and the magnetic charge of the black hole and has henceforth sometimes been called a secondary hair [5].

In effective four-dimensional string theory, however, further scalar fields are present in addition to the dilaton, such as, for example, the moduli coming from compactification of higher dimensions, which are nonminimally coupled to the Maxwell field [6]. The introduction of these fields may change the properties of the black holes. A simplified model which takes into account one modulus has been studied some time ago [7]. It was shown that an exact spherically symmetric black hole solution of the field equations can be found by requiring that the dilaton and the modulus are proportional. However, this restriction is not necessary, and it would be interesting to investigate the properties of the most general spherically symmetric solutions. In general, it is not possible to find these solutions in analytic form. (The field equations can in fact be cast in the form of a Toda molecule system of first order differential equations, which is exactly solvable only in a few special cases.) However, the qualitative behavior of the solutions and some quantitative results can be obtained by studying the Toda dynamical system. In particular, the metric and the scalar fields will necessarily be regular at all the points of the integral curves except critical points. Consequently, in order to determine the global properties of the solutions, such as the structure of their horizons and asymptotic regions, it suffices to study their behavior at the critical points of the dynamical system. One drawback of this method is that only the exterior region of the black hole can be studied. The interior may be, however, investigated numerically by continuing the solutions beside the horizon.

In this paper we undertake the investigation of the general solutions of the model introduced in Ref. [7] using this approach, and show that in general there exists a three-parameter family of asymptotically flat black hole solutions. This result is interesting because the third parameter can be presumably related to a scalar charge, giving therefore an example of primary scalar hair, in the sense of Ref. [5]. In addition to these solutions, the model also admits as a limiting case a two-parameter family of nonasymptotically flat black hole degenerate solutions of the kind discussed in Ref. [8]. We also investigate the properties of extremal black hole solutions, which are of great interest in recent developments of string and membrane theories.

The paper is organized as follows. In Sec. I we describe the model and obtain the dynamical system associated with the field equations. In Sec. II we discuss the exact black hole solutions, obtained for special values of the parameters. In Sec. III we study the dynamical system in its generality, while in Sec. IV we discuss the physical properties of its solutions.

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I. THE ACTION AND THE FIELD EQUATIONS

We study the action [7]

$$S=\int d^4x\sqrt{-g}\Big[R-2(\nabla\Phi)^2-\frac{2}{3}(\nabla\Sigma)^2 -(e^{-2\Phi}+\lambda^2 e^{-2q\Sigma/3})F^2\Big], \tag{1.1}$$

where Φ and Σ are the four-dimensional dilaton and modulus, respectively, F is the Maxwell field strength, and q and λ are coupling parameters. This action has been obtained by dimensional reduction of heterotic string effective action [9], with the addition of a nonminimal coupling term for the modulus, arising from integrating out heavy modes [6].

The field equations ensuing from Eq. (1.1) are

$$R_{\mu\nu}=2\nabla_\mu\Phi\nabla_\nu\Phi+\frac{2}{3}\nabla_\mu\Sigma\nabla_\nu\Sigma+2(e^{-2\Phi}+\lambda^2e^{-2q\Sigma/3})\times\left(F_\mu{}^\rho F_{\nu\rho}-\frac{1}{4}F^2g_{\mu\nu}\right),$$

$$\nabla^\mu[(e^{-2\Phi}+\lambda^2e^{-2q\Sigma/3})F_{\mu\nu}]=0,$$

$$\nabla^2\Phi=-\frac{1}{2}e^{-2\Phi}F^2,$$

$$\nabla^2\Sigma=-\frac{q\lambda^2}{2}e^{-2q\Sigma/3}F^2. \tag{1.2}$$

A magnetically charged spherically symmetric solution can be found assuming Maxwell field strength

$$F_{mn}=Q\epsilon_{mn},\quad m,n=2,3 \tag{1.3a}$$

and a metric of the form [1]

$$ds^2=e^{2\nu}(-dt^2+e^{4\rho}d\xi^2)+e^{2\rho}d\Omega^2, \tag{1.3b}$$

where ν, ρ, Φ, and Σ are functions of the ''radial'' coordinate ξ. Defining a new function $\zeta=\nu+\rho$, the field equations (2) take the simpler form

$$\zeta''=e^{2\zeta}, \tag{1.4a}$$

$$\Phi''=-Q_1^2e^{2\nu-2\Phi}, \tag{1.4b}$$

$$\Sigma''=-qQ_2^2e^{2\nu-2q\Sigma/3}, \tag{1.4c}$$

$$\nu''=Q_1^2e^{2\nu-2\Phi}+Q_2^2e^{2\nu-2q\Sigma/3} \tag{1.4d}$$

with $Q_1^2=Q^2$ and $Q_2^2=\lambda^2Q^2$, subject to the constraint

$$\zeta'^2-\nu'^2-\Phi'^2-\frac{1}{3}\Sigma'^2+Q_1^2e^{2\nu-2\Phi}+Q_2^2e^{2\nu-2q\Sigma/3}-e^{2\zeta}=0, \tag{1.5}$$

A first integral of Eq. (1.4a) is given by

$$\zeta'^2=e^{2\zeta}+a^2,$$

where a^2 is an integration constant, which has been chosen to be non-negative because otherwise one would obtain solutions with no asymptotic region, which are not of interest to us. For the moment we consider only strictly positive values of a. As we shall see, the limit $a\to 0$, corresponds to extremal solutions. Integrating again, with a suitable choice of the origin of ξ, one gets

$$e^{\zeta}=\frac{2ae^{a\xi}}{1-e^{2a\xi}}, \tag{1.6}$$

where a can be chosen to be positive without loss of generality. Moreover, from the remaining Eqs. (1.4), one obtains the relation

$$\frac{1}{q}\Sigma''+\Phi''+\nu''=0$$

which can be integrated to read

$$\Sigma'=-q(\nu'+\Phi'+c) \tag{1.7}$$

with c an integration constant. In view of Eqs. (1.4) and (1.7), defining

$$\chi=\nu-\Phi,\qquad \eta=\nu-\frac{q}{3}\Sigma \tag{1.8}$$

the field equations can be put in the ''Toda molecule'' form

$$\chi''=2Q_1^2e^{2\chi}+Q_2^2e^{2\eta},$$

$$\eta''=Q_1^2e^{2\chi}+\frac{3+q^2}{3}Q_2^2e^{2\eta} \tag{1.9}$$

In terms of χ and η, the derivatives of the fields Φ, Σ, and ν are given by

$$\Phi'=\frac{3}{3+2q^2}\left(\eta'-\frac{3+q^2}{3}\chi'-\frac{q^2}{3}c\right),$$

$$\Sigma'=\frac{3q}{3+2q^2}(\chi'-2\eta'-c),$$

$$\nu'=\frac{3}{3+2q^2}\left(\eta'+\frac{q^2}{3}\chi'-\frac{q^2}{3}c\right), \tag{1.10}$$

and Eq. (1.5) can be written

$$a^2-\frac{3}{3+2q^2}\left[\frac{3+q^2}{3}\chi'^2+2\eta'^2-2\eta'\chi'+\frac{q^2}{3}c^2\right]+Q_1^2e^{2\chi}+Q_2^2e^{2\eta}=0. \tag{1.11}$$

Equations (1.9) with the constraint (1.11) can be solved exactly in a few special cases, which are reported in the following section.

In the general case, they can be recast in the form of a three-dimensional system of first-order differential equations. If we define the variables

$$X=\chi', \quad Y=\eta', \quad Z=|Q_2|e^{\eta}$$

then the constraint (1.10) can be considered as a definition of $e^{2\chi}$. Eliminating the term $e^{2\chi}$ from Eqs. (1.9), one obtains the system

$$X'=Z^2+2P(X,Y,Z),$$
$$Y'=\frac{3+q^2}{3}Z^2+P(X,Y,Z),$$
$$Z'=YZ, \tag{1.12}$$

where

$$P(X,Y,Z)=Q_1^2e^{2\chi}=\frac{1}{3+2q^2}[(3+q^2)X^2+6Y^2-6XY-3B]-Z^2 \tag{1.13}$$

with $B=[(3+2q^2)/3]a^2-(q^2/3)c^2$.

II. EXACT SOLUTIONS

A. The $Q_2=0$ case

This limit case corresponds to minimal coupling of Σ, i.e., $\lambda\to 0$. By the no-hair theorem, the regular solutions should have constant Σ, as we shall verify. When $Q_2=0$, Eq. (1.9) take the form

$$\chi''=2Q_1^2e^{2\chi}, \quad \eta''=Q_1^2e^{2\chi}, \tag{2.1}$$

The first equation can be integrated to give

$$\chi'^2=2Q_1^2e^{2\chi}+b^2 \tag{2.2}$$

with b an integration constant. Moreover, comparing the two equations (2.1),

$$\eta'=\frac{1}{2}(\chi'-k) \tag{2.3}$$

with k an arbitrary constant. The constraint equation (1.11) then becomes

$$a^2-\frac{b^2}{2}-\frac{3}{3+2q^2}\left[\frac{1}{2}k^2+\frac{q^2}{3}c^2\right]=0 \tag{2.4}$$

Integrating again Eq. (2.2), one gets

$$Q_1e^{\chi}=\frac{\sqrt{2}bAe^{b\xi}}{1-A^2e^{2b\xi}} \tag{2.5}$$

with A an integration constant.

From these results and the relations (1.10), one can now write down the general solution in terms of the physical fields. Rather than giving all the explicit expressions, let us first consider the ''radial'' metric function $e^{\rho}=e^{\zeta-\nu}$. As $\xi\to 0$, $e^{\rho}\to\infty$, and hence one can identify this limit with spatial infinity. As $\xi\to-\infty$, instead, one has from Eqs. (1.6), (1.10), and (2.5),

$$e^{\rho}\sim\text{const}\times\exp\left\{\left[a-\frac{b}{2}+\frac{3}{3+2q^2}\left(\frac{1}{2}k+\frac{q^2}{3}c\right)\right]\xi\right\} \tag{2.6}$$

which implies that for $\xi\to-\infty$, $e^{\rho}\to 0$, giving rise to a singularity, except in the special case when the constant factor in the exponential vanishes, in which case $e^{\rho}\to$const as $\xi\to-\infty$. In conjunction with Eq. (2.4), this request singles out a unique real solution for the parameters, given by $a=b=-c=-k$.

In order to analyze the metric, it is useful to write it in a Schwarzschild-like form, by introducing a new radial coordinate r, such that $dr=e^{2\zeta}d\xi$. In the new coordinates,

$$ds^2=-e^{2\nu}dt^2+e^{-2\nu}dr^2+e^{2\rho}d\Omega^2, \tag{2.7}$$

where the metric functions are now viewed as functions of r. With a suitable choice of the origin of r, one then has

$$e^{2a\xi}=\frac{r-r_+}{r-r_-}, \quad e^{2\zeta}=(r-r_+)(r-r_-),$$
$$1-A^2e^{2a\xi}=(1-A^2)\frac{r}{r-r_-} \tag{2.8}$$

with $r_+=2a/(1-A^2)$, $r_-=2aA^2/(1-A^2)$. Moreover, if one chooses A such that $Q_1=2aA/(1-A^2)$, the physical fields read, in terms of the new radial coordinate,

$$e^{2\nu}=1-\frac{r_+}{r}, \quad e^{2\rho}=r^2\left(1-\frac{r_-}{r}\right),$$
$$e^{-2\Phi}=1-\frac{r_-}{r}, \quad e^{-2\Sigma}=\text{const}. \tag{2.9}$$

This is nothing but the well-known Garfinkle-Horowitz-Strominger (GHS) solution [1–2]. It describes asymptotically flat black holes with mass $r_+/2$ and charge $Q_1^2=r_+r_-/2$. The surface $r=r_+$ is a horizon while the point $r=r_-$ is a singularity.

Qualitatively different solutions arise in the special case $A=1$. In this case, $e^{2\zeta}\sim e^{2\chi}$, and choosing the origin of r such that $r_+=2a$, one gets

$$e^{2\nu}=r-r_+, \quad e^{2\rho}=r,$$
$$e^{-2\Phi}=r, \quad e^{-2\Sigma}=\text{const}. \tag{2.10}$$

This solution is not asymptotically flat, but still possesses a horizon at r_+ and is singular at the origin. It has been investigated in detail in Ref. [8].

Another important limit is reached when $a=0$ and corresponds to external black holes with $r_-=r_+$. In fact, in that case $e^{2\zeta}=\xi^{-2}$, and proceeding as before one can show that the existence of a regular horizon implies that also the parameters b, c, and k must vanish. Hence, one has $e^{2\chi}=(\xi+A)^{-2}$, with A an integration constant. Defining a new coordinate $r=A(1-A\xi^{-1})$, the metric functions can be finally cast in the form (2.9), with $r_+=r_-=A$.

B. The $Q_1=0$ case

This case corresponds to minimal coupling of Φ and can be considered the limit of Eq. (1.1) for $\lambda\to\infty$. By the no-hair theorem, the regular solutions must have constant Φ. The field equations are now

$$\chi''=Q_2^2e^{2\eta}, \quad \eta''=\frac{3+q^2}{3}Q_2^2e^{2\eta}. \tag{2.11}$$

Proceedings as before, one gets

$$Q_2e^{\eta}=\sqrt{\frac{3}{3+q^2}}\,\frac{2bAe^{b\xi}}{1-A^2e^{2b\xi}},$$

$$\chi'=\frac{3}{3+q^2}(\eta'-k), \tag{2.12}$$

where b, A, k are integration constants, together with the constraint

$$a^2-\frac{3}{3+q^2}b^2-\frac{3}{3+2q^2}\left[\frac{3}{3+q^2}k^2+\frac{q^2}{3}c^2\right]=0. \tag{2.13}$$

We again look for regular black hole solutions. For this purpose, we consider the asymptotic behavior of e^{ρ} as $\xi\to-\infty$, which is now

$$e^{\rho}\sim\text{const}\times\exp\left\{\left[a-\frac{3}{3+q^2}b+\frac{3}{3+2q^2}\left(\frac{q^2}{3+q^2}k+\frac{q^2}{3}c\right)\right]\xi\right\}. \tag{2.14}$$

A horizon can only occur when the coefficient of ξ in the exponential vanishes, in which case $e^{\rho}\to$const as $\xi\to-\infty$. This condition, together with Eq. (2.13) implies that $a=b=-c=-3k/q^2$.

Defining a new coordinate r as before, for A such that

$$Q_2=\sqrt{\frac{3}{3+q^2}}\,\frac{2aA}{1-A^2},$$

one gets finally

$$e^{2\nu}=\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{(3-q^2)/(3+q^2)},$$

$$e^{2\rho}=r^2\left(1-\frac{r_-}{r}\right)^{2q^2/(3+q^2)},$$

$$e^{-2\Phi}=\text{const}, \quad e^{-2\Sigma}=\left(1-\frac{r_-}{r}\right)^{6q/(3+q^2)}. \tag{2.15}$$

These solutions have not been considered previously, but essentially coincide with the generalized GHS solutions [1–2], where now Σ plays the role of the dilaton. They describe asymptotically flat black holes with mass $\frac{1}{2}\{r_++[(3-q^2)/(3+q^2)]r_-\}$ and charge $Q_2^2=[3/(3+q^2)]r_+r_-$. A horizon occurs at $r=r_+$ and a singularity at $r=r_-$. The external limit $r_+=r_-$ is achieved when $a=b=c=k=0$.

Also the limit $A=1$ is special and describes nonasymptotically flat black holes. For $A=1$ one has, in fact,

$$e^{2\nu}=(r-r_+)r^{(3-q^2)/(3+q^2)}, \quad e^{2\rho}=r^{2q^2/(3+q^2)},$$

$$e^{-2\Phi}=\text{const}, \quad e^{-2\Sigma}=r^{6q/(3+q^2)}. \tag{2.16}$$

Metrics of this form have been investigated in Ref. [8].

C. The case $\chi'=\eta'$

The last case in which exact solutions can be obtained is given by the condition $\eta=\chi+$const, which corresponds to the solutions found in Ref. [7]. Setting $e^{2\eta}=K^2e^{2\chi}$, the field equations become

$$\chi''=(2Q_1^2+K^2Q_2^2)e^{2\chi}=\left(Q_1^2+\frac{3+q^2}{3}K^2Q_2^2\right)e^{2\chi}. \tag{2.17}$$

Hence, $K^2=(3/q^2)(Q_1^2/Q_2^2)=3/\lambda^2q^2$, and

$$Q_1e^{\chi}=\sqrt{\frac{q^2}{3+2q^2}}\,\frac{2bAe^{b\xi}}{1-A^2e^{2b\xi}}, \tag{2.18}$$

where b and A are integration constants. The constraint (1.5) reduces to

$$a^2-\frac{3+q^2}{3+2q^2}b^2-\frac{q^2}{3+2q^2}c^2=0. \tag{2.19}$$

The solution possesses a horizon if

$$a-\frac{3+q^2}{3+2q^2}b+\frac{q^2}{3+2q^2}c=0. \tag{2.20}$$

From Eqs. (2.19) and (2.20), one obtains $a=b=-c$.

In terms of the coordinate r defined above, choosing A such that

$$Q_1=\sqrt{\frac{q^2}{3+2q^2}}\,\frac{2aA}{1-A^2},$$

the metric functions read

$$e^{2\nu}=\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{3/(3+2q^2)},$$

$$e^{2\rho}=r^2\left(1-\frac{r_-}{r}\right)^{2q^2/(3+2q^2)},$$

$$e^{-2\Phi}=\left(1-\frac{r_-}{r}\right)^{2q^2/(3+2q^2)},$$

$$e^{-2\Sigma}=(3/q^2\lambda^2)^{3/q}\left(1-\frac{r_-}{r}\right)^{6q/(3+2q^2)}. \quad (2.21)$$

(Notice that we have exchanged the definition of r_+ and r_-.) These solutions describe asymptotically flat black holes of mass

$$\frac{1}{2}\left(r_+ + \frac{3}{3+2q^2}r_-\right)$$

and charge [7]

$$Q_1^2=\frac{q^2}{3+2q^2}r_+r_-.$$

Also in this case the extremal black holes are obtained for vanishing a, b, and c.

In the special case $A=1$, the solutions reduce to

$$e^{2\nu}=(r-r_+)r^{3/(3+2q^2)},\quad e^{2\rho}=r^{2q^2/(3+2q^2)},$$

$$e^{-2\Phi}=r^{2q^2/(3+2q^2)},\quad e^{-2\Sigma}=(3/q^2\lambda^2)^{3/q}r^{6q/(3+q^2)} \quad (2.22)$$

and describe nonasymptotically flat black holes.

III. THE DYNAMICAL SYSTEM

The dynamical system (1.12) is similar to analogous systems studied in several contexts [10], but differs from these because the critical points at finite distance lie on a compact curve. It is easy to see, in fact, that all the critical points at finite distance are placed at the intersection between the plane $Z=0$ and the hyperboloid $P=0$, with P defined in Eq. (1.13), which (except in some degenerate cases) is an ellipse.

In particular, the plane $Z=0$ corresponds to the limit $Q_1=0$. The system is invariant under $Z\to -Z$, but the $Z<0$ half-space is simply a copy of the positive Z half-space and has no physical significance. Hence, we shall not consider it in the following.

The hyperboloid $P=0$ contains the trajectories corresponding to the limit $Q_2=0$. If $B>0$ it is one sheeted, while it is two sheeted if $B<0$. We shall consider only the former case. It is easy to see, in fact, that when $B<0$, the hyperboloid does not intersect the plane $Z=0$ and therefore there are no critical points at finite distance. It follows that the solutions of the dynamical system are of oscillatory type, and do not lead to reasonable black hole geometries. Moreover, the physically relevant solutions are those in the exterior of the hyperboloid, which corresponds to $|Q_2|e^{\chi}>0$, i.e., to the external region of the black hole. Finally, we notice that in the limit $B=0$, the hyperboloid reduces to a cone and the only critical point at finite distance is the origin of the coordinates. This limit corresponds to extremal black hole solutions.

As noted above, when $B>0$, the intersection of the hyperboloid $P=0$ with the plane $Z=0$ is given by an ellipse. More precisely, for every

$$|X_0|\leq\sqrt{\frac{9B}{(3+2q^2)(3+q^2)}}$$

there is a critical point at $X=X_0$, $Y=Y_0$, $Z=0$, where Y_0 is given in terms of X_0 by the solution of the quadratic equation

$$(3+q^2)X_0^2+6Y_0^2-6X_0Y_0-3B=0. \quad (3.1)$$

The characteristic equation for small perturbations,

$$X=X_0+x,\quad |x|\ll 1,$$

$$Y=Y_0+y,\quad |y|\ll 1,$$

$$Z=Z_0+z,\quad |z|\ll 1,$$

has eigenvalues 0, $2X_0$, and Y_0. Hence, each point in the $Z=0$ plane satisfying Eq. (3.1) with $X_0>0$, $Y_0>0$, repels a two-dimensional bunch of solutions in the full three-dimensional phase space, while solutions of Eq. (3.1) with $X_0<0$, $Y_0<0$ are attractors. The points with $X_0>0$, $Y_0<0$ or $X_0<0$, $Y_0>0$ act as saddle points. The presence of a vanishing eigenvalue is due of course to the fact that there is a continuous set of critical points lying on a curve. The critical points correspond to $\xi\to-\infty$ for trajectories starting from the ellipse, and to the limit $\xi\to\infty$ for trajectories ending at the ellipse.

The pattern of trajectories in the $Z=0$ plane, which correspond to the exact solutions discussed in Sec. II A is depicted in Fig. 1(a). They are given by lines of equation $Y=\frac{1}{2}(X-k)$, with k a constant. The lines which do not intersect the ellipse, i.e., those with $|k|>\sqrt{B}$, correspond to oscillatory behavior of X and Y and are not of interest to us. Notice that the extremal trajectories, for which $|k|=\sqrt{B}$ are tangent to the ellipse at $X=0$.

The projection of the trajectories corresponding to the exact solutions of Sec. II B from the hyperboloid to the $Z=0$ plane presents a similar phase portrait. The trajectories satisfy in this case the equation $Y=[(3+q^2)/3](X-k')$ and only trajectories with $|k'|<\sqrt{B/3}$ cross the ellipse. The projections of the extremal trajectories corresponding to $|k'|=\sqrt{B/3}$ are now tangent to the ellipse at $Y=0$ [see Fig. 1(b)].

For completeness, we notice that the solutions of Sec. 2 C are given by the hyperbola of equation

$$(3+q^2)(3+2q^2)Z^2-3(3+q^2)X^2=-9B, \quad (3.2)$$

lying in the plane $X=Y$.

We pass now to consider the behavior of the metric functions e^{ρ},e^{ν}, for $\xi\to-\infty$. In this limit, $e^{2\chi}\sim e^{2X_0\xi}$, $e^{2\eta}\sim e^{2Y_0\xi}$, and hence,

$$e^{\rho}\sim\exp\left[\frac{1}{3+2q^2}[(3+2q^2)a+q^2c-q^2X_0-3Y_0]\xi\right],$$

$$e^{2\nu}\sim\exp\left[\frac{2}{3+2q^2}(-q^2c+q^2X_0+3Y_0)\xi\right]. \quad (3.3)$$

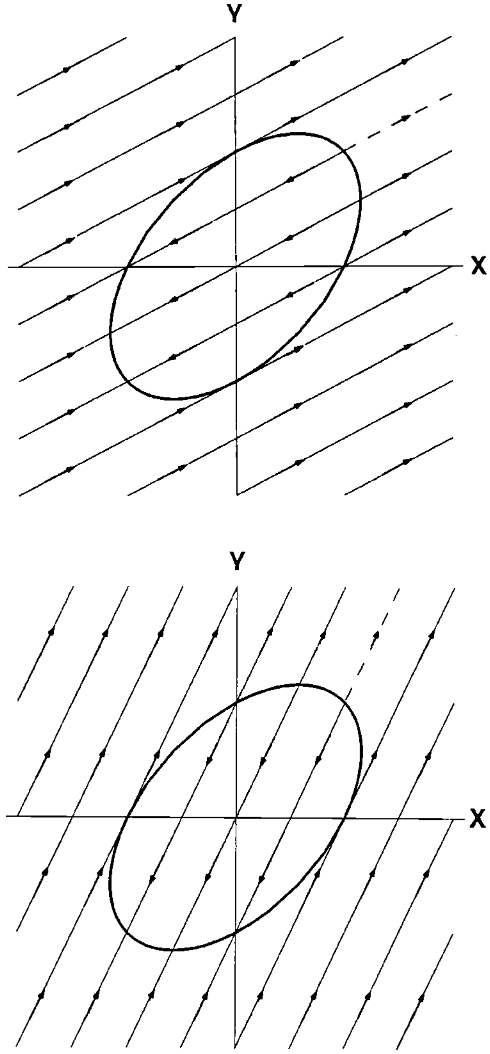

FIG. 1. (a) The phase plane $Z=0$. As discussed in the text, only the dashed trajectory corresponds to a regular black hole solution, while the others correspond to solutions with naked singularities. (b) The projections on the plane $Z=0$ of the trajectories lying on the hyperboloid. Also in this case, only the dashed line corresponds to a regular black hole solution.

In general, the radius $e^{\rho}\to 0$ as $\xi\to-\infty$, except in the special case

$$(3+2q^2)a+q^2c-q^2X_0-3Y_0=0. \tag{3.4}$$

This equation, combined with Eq. (3.1), gives the only real solution $X_0=Y_0=a=-c$. In this case $e^{\rho}\to$const as $\xi\to-\infty$. When these conditions are not satisfied, the metric function $e^{2\nu}$ is singular near the critical points, giving rise to a singularity as $e^{\rho}\to 0$.

Also when the relation (3.4) is satisfied, the metric function $e^{2\nu}$ behaves singularly near the critical points, but this can be shown to be simply a coordinate singularity by computing the curvature invariants, which tend to a constant value as $\xi\to-\infty$ when Eq. (3.4) holds. One can also check that, under these conditions, the scalar fields are regular for $\xi\to-\infty$. Therefore, all the trajectories starting from the point

$$X_0=Y_0=\sqrt{\frac{3}{3+q^2}B}$$

correspond to solutions with regular horizon, provided $c=-a$. These special trajectories are represented in Figs. 1(a) and 1(b) by dashed lines.

To complete the analysis of the phase space we must also investigate the nature of the critical points on the surface at infinity. This can be done by defining new coordinates u, y, and z such that infinity corresponds to $u\to 0$:

$$u=\frac{1}{X},\quad y=\frac{Y}{X},\quad z=\frac{Z}{X}.$$

Then Eqs. (1.12) take the form

$$\dot{u}=-(z^2+2p)u,$$

$$\dot{y}=-(z^2+2p)y+\frac{3+q^2}{3}z^2+p,$$

$$\dot{z}=(y-z^2-2p)z \tag{3.5}$$

where we have defined $p=P/X^2$ and a overdot denotes $ud/d\xi$. The critical points with $u=0$ can be classified in three categories.

(1) Two critical points, which we denote $L_{1,2}$ placed at $y=1/2$, $z=0$, i.e.,

$$X=\pm\infty,\quad Y=\frac{X}{2},\quad Z=0.$$

These are the end points of the trajectories lying in the $Z=0$ plane. The analysis of stability shows that the point with $X>0$ $(X<0)$ acts as an attractor (repellor) both on the trajectories coming from finite distance and on the two-dimensional bunch of trajectories lying on the surface at infinity.

(2) Two critical points $M_{1,2}$ lie at $y=(3+q^2)/3$, $z^2=(3+q^2)/3$, i.e.,

$$X=\pm\infty,\quad Y=\frac{3+q^2}{3}X,\quad Z=\sqrt{\frac{3+q^2}{3}}X.$$

These are the endpoints of the trajectories lying on the hyperboloid $P=0$. The analysis of stability shows that also in this case the point with $X>0$ $(X<0)$ attracts (repels) both the trajectories coming from finite distance and those lying on the surface at infinity.

(3) Two critical points $N_{1,2}$ lie at $y=1$, $z^2=3/(3+2q^2)$, i.e.,

$$X=\pm\infty,\quad Y=X,\quad Z=\sqrt{\frac{3}{3+2q^2}}X.$$

These are the end points of the hyperbola (3.2) in the $X=Y$ plane. The points with $X>0 (X<0)$ act as attractors (repellors) on the trajectories coming from finite distance and as saddle points on the trajectories at infinity.

In Fig. 2 we sketch the pattern of trajectories on the surface at infinity. The point at infinity is reached for $\xi\to\xi_0$,

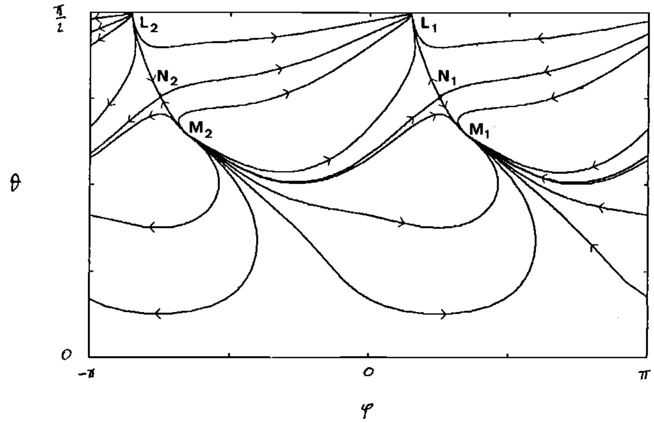

FIG. 2. The phase space on the sphere at infinity, where $\varphi = \arctan(Y/X)$, $\theta = \arctan(\sqrt{X^2+Y^2}/Z)$.

where ξ_0 is a finite constant. It is easy to see that for $\xi \to \xi_0$, the functions χ and η behave as

$$e^{\chi} \sim |\xi - \xi_0|^{1/v_0}, \quad e^{\eta} \sim |\xi - \xi_0|^{y_0/v_0},$$

where $v_0 \equiv z_0^2 + 2p_0$, the subscript 0 indicating the value taken at the critical points. Hence, if $\xi_0 \neq 0$, for $\xi \to \xi_0$, the metric functions behave as

$$e^{\nu} \sim e^{-\rho} \sim |\xi - \xi_0|^{[3/(3+2q^2)][y_0+q^2/3]v_0^{-1}},$$

$$e^{\Phi} \sim |\xi - \xi_0|^{[3/(3+2q^2)][y_0-(3+q^2)/3]v_0^{-1}},$$

$$e^{\Sigma} \sim |\xi - \xi_0|^{[3q/(3+2q^2)](1-2y_0)v_0^{-1}}. \tag{3.6}$$

The following picture of the phase space emerges: a family of trajectories start at the ellipse and end at one of the critical points L_1, M_1, N_1. Another family of trajectories start at one of the critical points L_2, M_2, or N_2 and end at the ellipse. Moreover, there are trajectories which never intersect the ellipse, connecting the critical points at $X = -\infty$ to those at $X = +\infty$. Of all the trajectories, only those starting at points of the ellipse such that $X_0 = Y_0$ can correspond to regular solutions. For completeness, we observe that most of the trajectories lying in the interior of the hyperboloid join M_1 to M_2, but we shall not study them in detail because they are devoid of physical significance.

IV. DISCUSSION

We finally discuss the implications of the phase space portrait of the previous section on the physical properties of the solutions. For this purpose, it is useful to define a new radial coordinate r such that $dr = e^{2\zeta} d\xi$, as in Sec. II. One has

$$r = \frac{r_+ - r_- e^{2a\xi}}{1 - e^{2a\xi}}, \tag{4.1}$$

where we have defined $r_+ = 2a(1 - e^{-2a\xi_0})^{-1}$, $r_- = 2ae^{-2a\xi_0}(1 - e^{-2a\xi_0})^{-1}$. In this way it is easy to identify the range of variation of ξ with the corresponding physical regions of the spacetime.

For $\xi \to \mp\infty$, $r \to r_\pm$, while for $\xi \to 0$, $r \to \infty$. Moreover, for $\xi \to \xi_0$, which without loss of generality we shall assume non-negative, $r \to 0$, except when ξ_0 vanishes.

If $\xi_0 \neq 0$, we can identify the trajectories starting at the ellipse and ending at the point ξ_0 with the exterior region of the black hole $r > r_+$. If the condition (3.4) is satisfied, these solutions possess a regular horizon. Moreover, they are asymptotically flat, since $e^{2\chi}$ and $e^{2\eta}$ tend to a constant as $\xi \to 0$. One can calculate the behavior of these solutions as $r \to r_+$ From Eqs. (3.3) and (3.4), one sees that for regular solutions $e^{2\nu} \sim e^{2a\xi}$ and hence, $e^{2\nu} \sim (r - r_+)$ for $r \to r_+$. In the same way one can see that the scalar fields are constant in that limit. It may be noticed that Eq. (3.6) implies that in the unphysical limit $r \to 0$, $e^{2\nu} \sim e^{-2\rho}$.

With our conventions, the trajectories starting at $X = -\infty$ and ending at the ellipse correspond to the unphysical region $0 < r < r_-$. Unfortunately, since r_- is in general a singularity, one cannot single out the trajectories corresponding to physical solutions by requiring the regularity of the curvature invariants near that point, as for r_+. Moreover, with our methods, we are not able to connect the solutions in the region $r > r_+$ with those in $r < r_+$ and then to discuss their behavior at $r = r_-$. This may, however, be achieved by using numerical methods.

The case $\xi_0 = 0$ needs a separate discussion. The solutions are no longer asymptotically flat, but their behavior for $r \to \infty$ can be obtained from the $\xi \to 0$ limit, which in our case turns out to be

$$e^{\nu} \sim |\xi|^{[3/(3+2q^2)](y_0+q^2/3)v_0^{-1}},$$

$$e^{\rho} \sim |\xi|^{1-[3/(3+2q)](y_0+q^2/3)v_0^{-1}},$$

$$e^{\Phi} \sim |\xi|^{[3/(3+2q^2)][y_0-(3+q^2)/3]v_0^{-1}},$$

$$e^{\Sigma} \sim |\xi|^{[3q/(3+2q)](1-2y_0)v_0^{-1}}.$$

Moreover, since for $\xi \to 0$, $r \sim |\xi|^{-1}$, it follows that for $r \to \infty$ the solutions behave in one of the following three ways, depending on the critical points where they terminate:

	$e^{2\nu}$,	$e^{2\rho}$,	$e^{-2\Phi}$,	$e^{-2\Sigma}$,
$L_{1,2}$,	r,	r,	r,	const,
$M_{1,2}$,	$r^{6/(3+q^2)}$,	$r^{2q^2/(3+q^2)}$,	const,	$r^{6q/(3+q^2)}$,
$N_{1,2}$,	$r^{(6+2q^2)/(3+2q^2)}$,	$r^{2q^2/(3+2q^2)}$,	$r^{2q^2/(3+2q^2)}$,	$r^{6q/(3+2q^2)}$.

These patterns coincide with those of the exact solutions (2.10), (2.16), or (2.22): hence all solutions of the system (1.2) are either asymptotically flat or possess the same asymptotic behavior as one of the exact nonflat solutions. Moreover, from the discussion of Sec. III of the phase space at infinity, it follows that the points $N_{1,2}$ are unstable, so that most trajectories of this class actually behave like the solutions (2.10) or (2.16) for $r\to\infty$.

As noticed above, the other relevant limit case, $B=0$, corresponds to the extremal black hole limit. In this case, the only critical point at finite distance is the origin of coordinates, and all the eigenvalues of the linearized equations vanish. This degeneration corresponds to a power-law behavior of the variables X, Y, and Z near the critical point: $X\sim -\alpha\xi^{-1}$, $y\sim -\beta\xi^{-1}$, $Z\sim\xi^{-\beta}$, $\sqrt{P}\sim\xi^{-\alpha}$. One can easily see from the field equations that the only possible values for α and β are $\alpha=1$, $\beta=1/2$, $\alpha=1$, $\beta=1$, and $\alpha=3/(3+q^2)$, $\beta=1$, which coincide with those of the exact extremal solutions of Sec. II. One can also check numerically that only the values $\alpha=1$, $\beta=1$ are stable, so that all the trajectories, except the exact ones, behave near the critical point at the origin similar to the solutions (2.21) (case C). This is interesting, because from the previous discussion we know that this limit corresponds to the horizon of the extremal black hole. Now, it is well known that in the cases A and C of Sec. II, the extremal ''string'' metric $d\hat{s}^2=e^{2\phi}ds^2$ has a ''near-horizon'' limit in which the metric function $e^{2\rho}$ becomes constant [11], and hence the spacetime decouples in the direct product of two two-dimensional spaces, while this is not true for case B. But since all solutions except A and B, behave as C near the horizon, we can conclude that solution B is the only one for which $e^{2\rho}$ is not constant near the horizon.

Before concluding this section, it is important to remark that the qualitative properties of the phase space and hence of the solutions are unaffected by the value of the parameter q, which is therefore essentially irrelevant for our discussion.

V. CONCLUSIONS

From the previous discussion results that there is a large class of asymptotically flat regular black hole solutions of the field equations (1.2). These are characterized by three parameters: mass, magnetic charge (or equivalently r_+ and r_-, or a and ξ_0), and a third parameter which classifies the different trajectories starting from the critical points $X_0=Y_0=\sqrt{[3/(3+q^2)]B}$, $Z_0=0$. We conjecture that the third parameter can be related to (a combination of) the scalar charges of the dilaton and the modulus. This conjecture cannot be checked explicitly because only in a few special cases the solution can be written in an analytic form.

The presence of an independent scalar charge would represent a novelty in the context of the no-hair results. In fact, in the known cases of dilaton gravity with nonminimal dilaton-Maxwell coupling, even if the dilaton is nontrivial, its charge is not an independent parameter, but is related to the mass and magnetic charge of the black hole (secondary hair). In our case of two nonminimally coupled scalar fields, it seems instead that a new independent charge is needed in order to classify the solutions.

Another interesting result is that in the extremal limit all the solutions except the unphysical case of a minimally coupled dilaton, have the same behavior near the horizon, decoupling into the product of two two-dimensional spaces. This is interesting since such a behavior is required in recent attempts of calculating black hole entropy by counting microstates of a conformal field theory [12].

Finally, we have clarified the role of non-asymptotically flat solutions, which were first discussed in Ref. [8] in the case of ordinary dilaton gravity, and shown that in our model they form a two-parameter family whose asymptotic behavior can assume only three possible forms.

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